PROBABILITY AND RANDOM PROCESSES

UNIT - I RANDOM VARIABLES

PROBABILITY THEORY :

The Probability of an event is defined as

$$P = \frac{\text{No. of favourable cases}}{\text{Total no. of Exhaustive cases}}$$

Example :

* Every day Sun rises in the East
* It is possible to live without water
* Probably Arjun gets the Job.

In the above statement, there is certainity in the statement 1, Impossibility in the statement 2 and uncertainity in the statement 3.

In the probability theory, we represent certainity by '1', Impossibility by '0' and uncertainity by a positive fraction between 0 and 1.

Remark :

* If 'p' denotes the probability of success and 'q' denotes the probability of failure then

$p+q=1$. Also $q=1-p$

* Probability always lies between 0 and 1.

* Probability of sure event is given by

$$P(S) = 1$$

* Probability of an impossible event is given by $P(\phi) = 0$

Laws of Probability :

Addition Law of probability :

Case i:

If A and B are not mutually exclusive events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Case ii:

If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B) \qquad (\because P(A \cap B) = \phi)$$

MULTIPLICATION LAW OF PROBABILITY :

For the two events A and B

case 1: If A and B are independent then

$$P(A \cap B) = P(A) \cdot P(B)$$

Case 2: If A and B are dependent then

$$P(A \cap B) = P(A) \cdot P(B/A), \quad P(A) > 0$$

$$= P(B) \cdot P(A/B), \quad P(B) > 0$$

i.e., $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ and

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

NOTE:

* $P\{\text{atleast 'a' event}\} = P(x \geq a)$
* $P\{\text{atmost 'a' event}\} = P(x < a)$
* $P(x \geq a) = 1 - P(x < a)$

Random variable (or) Stochastic Variable:

A random variable is defined as a rule that assigns numerical value to each possible outcome of an random experiment.

It is also called as 'STOCHASTIC VARIABLE'.

Example:

Consider the experiment of tossing a coin twice and let x denotes the number of heads in the outcome.

Sample space	HH	HT	TH	TT
Random variable	2	1	1	0
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

TYPES OF RANDOM VARIABLE :

DISCRETE RANDOM VARIABLE :

If X is the random variable which can take a finite (or) countably infinite number of values, X is called as discrete random variable.

When X is a discrete RV, the values of X may be assumed as $x_1, x_2, \ldots$

Example :

* A number shown when a die is thrown.
* The number of alpha particles emitted by a radio active source.

CONTINUOUS RANDOM VARIABLE :

A random variable X is said to be continuous if it takes all the values in an interval which may be finite.

Example : Height, weight and Age of individuals.

PROBABILITY DISTRIBUTIONS :

PROBABILITY MASS FUNCTION (For Discrete Random Variable)

If X is a random variable which can takes the values $x_1, x_2, x_3, \ldots$ such that $P\{X = x_i\} = P_i$ is called Probability function or Probability mass function provided P_i satisfy the following conditions

* $0 \leq P_i \leq 1$

* $\sum_i P_i = 1$

CUMULATIVE DISTRIBUTION FUNCTION [CDF] - DISCRETE

(OR)

DISTRIBUTION FUNCTION

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P_i$$

PROBABILITY DENSITY FUNCTION (PDF) - CONTINUOUS

If X is a continuous Random variable such that

$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = f(x)$ which satisfies the following conditions.

* $f(x) \geq 0$ for all x in R

* $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Note :

$$P(X=a) = \int_a^a f(x)\,dx = 0$$

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

$$= \int_a^b f(x)\,dx$$

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

(or)

DISTRIBUTION FUNCTION – for Continuous RV

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

~~PROBABILITY~~

PROBERTIES of cumulative distribution function:

* $F(-\infty) = 0$
* $0 \leq F(x) \leq 1$
* $F(\infty) = 1$
* If $x_1 < x_2$ then $F(x_1) < F(x_2)$
* If $F(x)$ is CDF of X and if $a<b$ then
  $P(a<x<b) = F(b) - F(a)$

Relation between PDF and CDF:

If X is a random variable then ~~f(x)=~~

$$f(x) = \frac{d}{dx} F(x).$$

Problems :-

1. A random variable x has the following probability fun.

$X = x_i$	0	1	2	3	4	5	6	7
$P\{X=x_i\}$:	0	K	$2k$	$2k$	$3k$	K^2	$2k^2$	$7k^2+K$

(i) Find K (ii) Evaluate $P(X \geq 6)$, $P(X<6)$ (iii) $P(X \leq c) > \frac{1}{2}$, then find the minimum value of c (iv) Evaluate $P(1.5 < X < 4.5 / X > 2)$

(v) Find $P(X<2)$, $P(X>3)$, $P(1<X<5)$.

Solution :-

(i) To find K.

We know that $\Sigma P_i = 1$

$$\therefore \; 0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$$

$$10K^2 + 9K = 1$$

$$10K^2 + 9K - 1 = 0$$

$$(10K-1)(K+1) = 0$$

$$10K - 1 = 0; \quad K+1 = 0$$

$$10K = 1; \quad K = -1 \text{ (Not possible)}$$

$$K = \frac{1}{10};$$

$$\boxed{\therefore K = \frac{1}{10}}$$

X :	0	1	2	3	4	5	6	7	Tot
$P(X=x_i)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$	= 1

(ii) Evaluate $P(x<6)$

$$P(x<6) = 1 - P[x \geq 6]$$

$$= 1 - [P(x>6) + P(x>7)]$$

$$= 1 - \left[\frac{2}{100} + \frac{17}{100}\right]$$

$$= 1 - \frac{19}{100}$$

$$= \frac{100-19}{100}$$

$$\boxed{P(x<6) = \frac{81}{100}}$$

$$P(x \geq 6) = P(x>6) + P(x>7)$$

$$= \frac{2}{100} + \frac{17}{100}$$

$$\boxed{P(x \geq 6) = \frac{19}{100}}$$

(iii) We know that $P(X \leq x) = F(x)$.

x	$f(x)$	$F(x) = P(X \leq x)$
0	0	0
1	$\frac{1}{10}$	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{3}{10}$
3	$\frac{2}{10}$	$\frac{5}{10} = \frac{1}{2}$
4	$\frac{3}{10}$	$\frac{8}{10} > \frac{1}{2}$
5	$\frac{1}{100}$	$\frac{81}{100}$

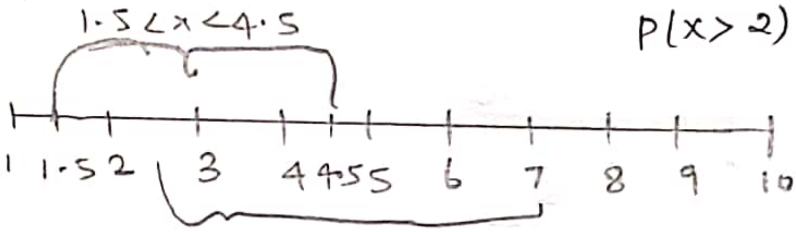
6	$\frac{2}{100}$	$\frac{83}{100}$
7	$\frac{17}{100}$	$\frac{100}{100} = 1$

The minimum value of c for which $P(x \le c) > \frac{1}{2}$ is 4.

(iv) Evaluate $P(1.5 < x < 4.5 / x > 2)$

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$P(1.5 < x < 4.5 / x > 2) = \frac{P(1.5 < x < 4.5 \cap x > 2)}{P(x > 2)}$$



$$P(1.5 < x < 4.5 | x > 2) = \frac{P(x=3) + P(x=4)}{1 - P(x \le 2)}$$

$$= \frac{P(x=3) + P(x=4)}{1 - [P(x=0) + P(x=1) + P(x=2)]}$$

$$= \frac{\frac{2}{10\cancel{0}} + \frac{3}{10\cancel{0}}}{1 - \left(0 + \frac{1}{10} + \frac{2}{10}\right)}$$

$$= \frac{\frac{5}{10}}{1 - \frac{3}{10}}$$

$$= \frac{\frac{5}{10}}{\frac{7}{10}} = \boxed{\frac{5}{7}}$$

(v) $P(X<2) = P(X=0) + P(X=1)$

$= 0 + \frac{1}{10}$

$$\boxed{P(X<2) = \frac{1}{10}}$$

$P(X>3) = 1 - P(X \le 3)$

$= 1 - \left(P(X=0) + P(X=1) + P(X=2) + P(X=3)\right)$

$= 1 - \left(0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10}\right)$

$= 1 - \frac{5}{10}$

$= \frac{5}{10}$

$$\boxed{P(X>3) = \frac{1}{2}}$$

$P(1<X<5) = P(X=2) + P(X=3) + P(X=4)$

$= \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$$\boxed{P(1<X<5) = \frac{7}{10}}$$

2. If X be a Random Variable with $E(X)=1$ and $E(X(X-1))=4$. Find $V(X)$ and $V(2-3X)$.

Solution:

Given $E(X)=1$, $E(X(X-1))=4$

$$E((X^2-X))=4$$

$$E(X^2)-E(X)=4$$

$$E(X^2)-1=4$$

$$\boxed{E(X^2)=5}$$

We know that $V(X)=E(X^2)-(E(X))^2$

$$=5-1$$

$$\boxed{V(X)=4}$$

Let $V(2-3X)=V(2)+9V(X)$

$$=0+9(4)$$

$$\boxed{V(2-3X)=36}$$

3. If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$. Find the Probability distribution and cumulative distribution function of X. Also find the mean and variance.

Solution:

Gn $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$

Let us take $\boxed{P(X=3)=K}$

Then $2P(X=1)=K$

$$\Rightarrow \boxed{P(X=1)=\frac{K}{2}}$$

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$3P(X=2) = k$

$$\boxed{P(X=2) = \frac{k}{3}}$$

$5P(X=4) = k$

$$\boxed{P(X=4) = \frac{k}{5}}$$

x :	1	2	3	4
$P(X=x)$:	$\frac{k}{2}$	$\frac{k}{3}$	k	$\frac{k}{5}$

We know that $\sum_i P_i = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$\boxed{k = \frac{30}{61}}$$

$\therefore$

x :	1	2	3	4
$P(X=x)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Cumulative distribution function (CDF)

$$F(x) = P(X \le x) = \sum_{x_i \le x} P_i$$

x	$P(X)$	$F(X)$
1	$\frac{15}{61}$	$\frac{15}{61}$
2	$\frac{10}{61}$	$\frac{25}{61}$
3 ~~4~~	$\frac{30}{61}$	$\frac{55}{61}$
4	$\frac{6}{61}$	$\frac{61}{61} = 1$

<u>Mean</u>

$$E(X) = \sum x P(X=x) = 1 \cdot \frac{15}{61} + 2 \cdot \frac{10}{61} + 3 \cdot \frac{30}{61} + 4 \cdot \frac{6}{61}$$

$$= \frac{15}{61} + \frac{20}{61} + \frac{90}{61} + \frac{24}{61}$$

$$\boxed{E(X) = \frac{149}{61}}$$

$$E(X^2) = \sum x^2 P(X=x) = 1^2 \cdot \frac{15}{61} + 2^2 \cdot \frac{10}{61} + 3^2 \cdot \frac{30}{61} + 4^2 \cdot \frac{6}{61}$$

$$E(X^2) = \frac{15}{61} + \frac{40}{61} + \frac{270}{61} + \frac{96}{61}$$

$$\boxed{E(X^2) = \frac{421}{61}}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{421}{61} - \left(\frac{149}{61}\right)^2 = 6.9 - 5.966$$

$$\boxed{V(X) = \cancel{\frac{272}{61}}} = 0.934$$

4. The Probability function of an infinite discrete distribution is given by $P(X=j) = \frac{1}{2^j}$, $j = 1, 2, 3, \ldots$. Find the mean and Variance of the distribution. Also find $P(X = \text{even})$, $P(X \text{ is } \div \text{ by } 3)$, $P(X \geq 5)$.

Solution :

Given $P(X=j) = \frac{1}{2^j}$, $j = 1, 2, \ldots$.

$X = j$	1	2	3	4	5 ....
$P(X=j)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$ ....

Mean:

$$E(X) = \sum x P(X=x)$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + 4 \cdot \frac{1}{2^4} + \ldots$$

$$= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{2}\left[1 + 2x + 3x^2 + 4x^3 + \ldots\right], \quad \text{where } x = \frac{1}{2}$$

$$= \frac{1}{2}(1-x)^{-2} = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{1}{2} \times 2^2 = \frac{4}{2} = 2$$

$$\boxed{E(X) = 2}$$

To find $E(X^2)$:

$$E(X^2) = \sum x^2 P(X=x) = \sum\left(x(x-1)+x\right) P(X=x)$$

$$= \sum_{x=1}^{\infty} x(x-1) P(X=x) + \sum x P(X=x)$$

$$= 0 + 2\cdot 1\cdot P(X=2) + 3\cdot 2\, P(X=3) + 4\cdot 3\cdot P(X=4) + \cdots\cdots + E(X)$$

$$= 2\,\frac{1}{2^2} + 3\cdot 2\cdot \frac{1}{2^3} + 4\cdot 3\,\frac{1}{2^4} + \cdots + 2$$

$$= 2\left(\frac{1}{2}\right)^2 + 2\cdot 3\cdot\left(\frac{1}{2}\right)^3 + 3\cdot 4\left(\frac{1}{2}\right)^4 + \cdots\cdots + 2$$

$$= \left(\frac{1}{2}\right)\left[2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2\right.$$

$$= 2\left(\frac{1}{2}\right)^2\left[1 + 3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + \cdots\right] + 2$$

$$= \frac{1}{2}\left(1-x\right)^{-3} + 2$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right)^{-3} + 2$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-3} + 2$$

$$E(X^2) = \frac{1}{2}\left(2^3\right) + 2$$

$$\boxed{E(X^2) = 6}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$V(X) = 6 - 2^2$$

$$\boxed{V(X) = 2}$$

$P(x = \text{even})$

$$P(x = \text{even}) = P(x=2) + P(x=4) + P(x=6) + \cdots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1}{2^2}\right) + \left(\frac{1}{2^2}\right)^2 + \cdots\right]$$

$$= \frac{1}{4}\left[1 + x + x^2 + \cdots\right], \quad x = \frac{1}{2^2} = \frac{1}{4}$$

$$= \frac{1}{4}(1-x)^{-1}$$

$$= \frac{1}{4}\left(1-\frac{1}{4}\right)^{-1}$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^{-1} = \frac{1}{4} \times \frac{4}{3}$$

$$\boxed{P(x = \text{even}) = \frac{1}{3}}$$

$P(x = \text{is} \div \text{by } 3) = P(x=3) + P(x=6) + P(x=9) + \cdots$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots$$

$$= \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots\right]$$

$$= \frac{1}{8}\left[1 + x + x^2 + \cdots\right], \quad x = \frac{1}{2^3} = \frac{1}{8}$$

$$= \frac{1}{8}\left(1 - \frac{1}{8}\right)^{-1} = \frac{1}{8}\left(\frac{7}{8}\right)^{-1} = \frac{1}{7}$$

$$P(X \geq 5) = 1 - P(X < 5)$$

$$= 1 - [P(X=1) + P(X=2) + P(X=3) + P(X=4)]$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)$$

$$= 1 - \left(\frac{8+4+2+1}{16}\right)$$

$$P(X \geq 5) = 1 - \frac{15}{16}$$

$$\boxed{P(X \geq 5) = \frac{1}{16}}$$

Problems on Continuous Random Variable

5. If the density function of a continuous random variable X is given by $f(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$

Find the value of 'a' and the cumulative distribution function.

Solution:

To find a:

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1$$

$$a\left(\frac{1}{2}-0\right) + a\,(2-1) + a\left(\left(9-\frac{9}{2}\right)-\left(6-\frac{4}{2}\right)\right) = 1$$

$$a\left[\frac{1}{2}+1+\frac{9}{2}-\frac{8}{2}\right] = 1$$

$$a\left[\frac{1}{2}+1+\frac{1}{2}\right] = 1$$

$$a\left[\frac{1+2+1}{2}\right] = 1$$

$$a\left(\frac{4}{2}\right) = 1$$

$$2a = 1$$

$$\boxed{a = \frac{1}{2}}$$

$$\therefore \quad f(x) = \begin{cases} \frac{x}{2} & ; \quad 0 \le x \le 1 \\ \frac{1}{2} & ; \quad 1 \le x \le 2 \\ \frac{3}{2} - \frac{x}{2} & ; \quad 2 \le x \le 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Cumulative Distribution Function :

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

In the Interval $0 \le x \le 1$

$$F(x) = \int_{-10}^{0} 0\,dx + \int_{0}^{x} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x$$

$$\boxed{F(x) = \frac{x^2}{4}}$$

In the Interval $1 \le x \le 2$

$$F(x) = \int_{-10}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0\right) + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$\boxed{F(x) = \frac{x}{2} - \frac{1}{4}}$$

In the Interval $2 \le x \le 3$

$$F(x) = \int_{-10}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^x$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0\right) + \frac{1}{2}(2-1) + \frac{1}{2}\left(3x - \frac{x^2}{2} - 6 + 2\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - \frac{4}{2}$$

$$\boxed{F(x) = \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}}$$

$$\frac{1}{4} + \frac{1}{2} - 2$$

$$\frac{1+2-8}{4}$$

In the Interval $x > 3$

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{2}dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx + \int_3^x 0\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^3 + 0$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(2-1) + \frac{1}{2}\left(\left(9 - \frac{9}{2}\right) - (6-2)\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\left(\frac{9}{2} - 4\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$$

$$F(x) = \frac{1+2+1}{4}$$

$$\boxed{F(x) = 1}$$

$$\therefore F(x) = \begin{cases} \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4}, & 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

6. A continuous random variable has the pdf of $f(x) = kx^4 \quad (-1 < x < 0)$. Find the value of k. Find $P\left[x > -\frac{1}{2} \mid x < -\frac{1}{4}\right]$

Solution: Gn $f(x) = kx^4 \quad (-1 < x < 0)$.

To find k:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-1}^{0} kx^4\,dx = 1$$

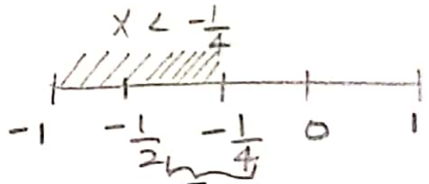
$$k\left[\frac{x^5}{5}\right]_{-1}^{0} = 1$$

$$k\left(0 - \left(\frac{-1}{5}\right)\right) = 1$$

$$k\left(\frac{1}{5}\right) = 1$$

$$k = 5$$

$$\boxed{f(x) = 5x^4, \quad -1 < x < 0}$$



(ii) $$P\left[x > -\frac{1}{2} \mid x < -\frac{1}{4}\right] = \frac{P\left[x > -\frac{1}{2} \cap x \leq -\frac{1}{4}\right]}{P\left[x < -\frac{1}{4}\right]} \quad x > -\frac{1}{2}$$

$$= \frac{P\left[-\frac{1}{2} \leq x \leq -\frac{1}{4}\right]}{P\left[x \leq -\frac{1}{4}\right]}$$

$$= \frac{\int_{-1/2}^{-1/4} 5x^4\,dx}{\int_{-1}^{-1/4} 5x^4\,dx}$$

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$$= \frac{\int_{-\frac{1}{2}}^{-1/4} x^4\, dx}{\int_{-1}^{-1/4} x^4\, dx}$$

$$= \frac{\left(\frac{x^5}{5}\right)_{-1/2}^{-1/4}}{\left(\frac{x^5}{5}\right)_{-1}^{-1/4}} = \frac{-\frac{1}{1024} + \frac{1}{32}}{-\frac{1}{1024} + 1}$$

$$= 0.03017$$

7. A continuous random variable X takes the values between $x = 2$ and $x = 5$ has the density function given by $f(x) = k(1+x)$. Find $P(X<4)$.

Solution:

Given $f(x) = k(1+x)$, $2 \le x \le 5$

To find k:

WKT $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{2}^{5} k(1+x)\, dx = 1$$

$$k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1$$

$$k\left[5 + \frac{25}{2} - 4\right] = 1$$

$$k\left[\frac{10+25-8}{2}\right]=1$$

$$k\left[\frac{27}{2}\right]=1$$

$$\boxed{k=\frac{2}{27}}$$

$$\therefore \quad f(x)=\frac{2}{27}(1+x), \quad 2\le x\le 5$$

<u>To find $P(x<4)$:</u>

$$P(x<4)=\int_{-\infty}^{4} f(x)\,dx=\int_{2}^{4}\frac{2}{27}(1+x)\,dx$$

$$=\frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^4$$

$$=\frac{2}{27}\left[(4+8)-(2+2)\right]$$

$$P(x<4)=\frac{2}{27}[8]$$

$$\boxed{P(x<4)=\frac{16}{27}}$$

8. The cumulative distribution function (CDF) of a random variable X is $F(x)=1-(1+x)e^{-x}$, $x>0$. Find the probability density function of X, Mean and variance.

Soln:

Gn $\quad F(x)=1-(1+x)e^{-x}, \quad x>0$

(i) To find PDF:

WKT $f(x) = \frac{d}{dx} F(x)$

$$= \frac{d}{dx}\left[1-(1+x)e^{-x}\right]$$

$$= -e^{-x} - (1+x)(-e^{-x})$$

$$= e^{-x}\left[-1+1+x\right]$$

$$\boxed{f(x) = xe^{-x}, \quad x>0}$$

(ii) Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x(xe^{-x})\,dx = \int_0^{\infty} x^2 e^{-x}\,dx$$

$u = x^2$ $\quad v = e^{-x}$

$u' = 2x$ $\quad v_1 = -e^{-x}$

$u'' = 2$ $\quad v_2 = e^{-x}$

$u''' = 0$ $\quad v_3 = -e^{-x}$

$+ \quad - \quad +$

$$E(X) = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$\boxed{E(X) = 2} \qquad \boxed{\because e^{-\infty} = 0, \; e^{0} = 1}$$

(iii) Variance:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2(xe^{-x})\,dx = \int_0^{\infty} x^3 e^{-x}\,dx$$

$u = x^3$ $\quad v = e^{-x}$

$u' = 3x^2$ $\quad v_1 = -e^{-x}$

$u'' = 6x$ $\quad v_2 = e^{-x}$

$u''' = 6$ $\quad v_3 = -e^{-x}$

$u^{iv} = 0$ $\quad v_4 = e^{-x}$

$\therefore \quad E(x^2) = \left[-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6 e^{-x} \right]_0^{\infty}$

$\boxed{E(x^2) = 6}$

$V(x) = E(x^2) - (E(x))^2$

$= 6 - 2^2$

$\boxed{V(x) = 2}$